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Parametric resonance in ideal magnetohydrodynamics

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We show that an external nonelectromagnetic periodic inhomogeneous force sets up a parametric resonance in an ideal magnetohydrodynamics. Alfven waves with certain wavelengths grow exponentially in amplitude. Nonlinear interaction between the resonant harmonics produces the long-term modulation of amplitudes. The mechanism of the energy transformation from an external nonelectromagnetic force to magnetic oscillations of the system presented here can be used in understanding the physical background of the gravitational action on the magnetized medium. Future application of this theory to several astrophysical problems is briefly discussed.

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I. INTRODUCTION

The stability of open systems is a principal problem in any branch of natural sciences. Action of an external force may create an inhomogeneity in a system and may result in an instability. Therefore, the type of the instability is defined by the character of the external action. For instance, the external force that results in a periodic change of the system parameter may generate the instability due to the parametric resonance. The most famous example of such a process may occur in the system of the mathematical pendulum with periodically varying length. The parametric resonance in plasma caused by an external electromagnetic force has been studied intensively (see, e.g., [1,2] and references therein). On the other hand, the investigation of the dynamics of the plasma that is externally affected by forces of a nonelectromagnetic origin is not well developed. In the present paper we study the influence of the external nonelectromagnetic action on the magnetohydrodynamic (MHD) system and anticipate its parametric form. We found that the external force, which is able to generate a periodic shear motion, could be responsible for the amplification of the magnetic fields due to a parametric resonance.

The externally generated periodic shear motions take place in many astrophysical objects: binary stars, stars with planetary systems, interacting galaxies, etc. Recently Zaqarashvili [3,4] has shown that the gravity of the planets results in a weak periodic shear in the internal rotation of the Sun. The latter causes the parametric amplification of the fossil magnetic field and leads to the formation of the solar cycles. Another example is the density waves in spiral galaxies that have been considered as an origin of the periodic change in the differential rotation [5,6], which sets up the resonance in the turbulent dynamo theory. At the same time, it appears that the consideration of the spiral density waves (originated from the gravitational and rotational actions) as an external force in the parametric form in the ideal MHD equations likely leads again to the parametric resonance for Alfven waves. In this case it turns out that the influence of one type of wave (density waves) on another (Alfven waves) may be studied in the linear approximation, despite of its nonlinear character. In general, the investigation of the physics of the external nonelectromagnetic periodic action on magnetized medium is very useful not only in theory, but also in the astrophysical context.

The external nonelectromagnetic periodic action on plasma may be characterized by the stability of the generated periodic inhomogeneous motion. Linearized MHD equations describing the stability of such equilibrium flow have both the spatial and temporal inhomogeneities. For this reason the classical stability theory leads to the set of partial differential equations that involves both spatial and temporal derivatives. However, rewriting the equations in comoving coordinates of the unperturbed flow [7–26] provides a way of retaining only the temporal inhomogeneity. Consequent Fourier expansion leads to the set with temporal derivatives only. This enables the analytical solution of MHD equations and thus the study of the physical background of the external action.

As already noted, periodic shear flow intuitively gives rise to the parametric instability for magnetic waves. This is another way of stating that the external nonelectromagnetic energy transforms into the magnetic energy of the system. Recent studies have revealed many problems in explaining the magnetic field behavior in the Sun, binary stars, galaxies, etc. [27,28]. Therefore, the presented mechanism of the magnetic field amplification may play a significant role in the solution of many astrophysical problems.

Section II contains the physical approach and mathematical formalism of the problem considered. Numerical simulations and formal nonlinear analysis are outlined in Secs. II A and II B, respectively. Discussion and applications to different astrophysical contexts (the Sun, binary stars,spiral galaxies) are presented in Sec. III.

II. MATHEMATICAL FORMALISM

To get a better insight into the nature of the parametric resonance in the ideal MHD we consider a boundary-free, homogeneous, magnetized medium. The medium considered is stable in the absence of an external action. However, this medium may became unstable when affected by an external force. Formally, the ideal MHD equations read as follows:

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{V}\times\mathbf{B}), \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right) = -\nabla p + \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B} + \mathbf{F}(r,t), \tag{2}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{3}$$

where ρ is the plasma density, p is the pressure, $\mathbf{V}$ is the velocity, $\mathbf{B}$ is the magnetic field, and $\mathbf{F}(r,t)$ is an external force of a nonelectromagnetic origin. For simplicity we assume the incompressibility $\rho = \text{const}$.

The X axis of the frame is directed along the unperturbed magnetic field,

$$\mathbf{B} = (B_{0,0,0}). \tag{4}$$

For simplicity the external force is directed along the magnetic field and is linearly inhomogeneous along the Y axis

$$\mathbf{F} = (\alpha \cos(\omega_0 t)y, 0, 0), \tag{5}$$

were $\omega_0 = 2\pi/t_0$ is the frequency of the external force. This action generates the periodic shear motion of plasma along the magnetic field lines having no effect on the field lines themselves. The velocity field of the generated flow takes the following form in the case of the homogeneous unperturbed pressure:

$$\mathbf{V}_0 = (V_{xy}y, 0, 0), \tag{6}$$

where

$$V_{xy}(t) = \frac{\alpha}{\rho\omega_0}\sin(\omega_0 t). \tag{7}$$

We use the linear perturbation theory to investigate the stability of such flow. All physical quantities are presented as the sum of the equilibrium and perturbed components,

$$\phi = \phi_0 + \phi'. \tag{8}$$

Then the linearization of Eqs. (1)–(3) leads to the following system:

$$\left[\frac{\partial}{\partial t} + (\mathbf{V}_0\cdot\nabla)\right]\mathbf{B}' = (\mathbf{B}_0\cdot\nabla)\mathbf{V}' + (\mathbf{B}'\cdot\nabla)\mathbf{V}_0, \tag{9}$$

$$\nabla\cdot\mathbf{B}' = 0, \tag{10}$$

$$\left[\frac{\partial}{\partial t} + (\mathbf{V}_0\cdot\nabla)\right]\mathbf{V}' = -\frac{\nabla P'}{\rho} + \frac{(\mathbf{B}_0\cdot\nabla)\mathbf{B}'}{4\pi\rho} - (\mathbf{V}'\cdot\nabla)\mathbf{V}_0, \tag{11}$$

$$\nabla\cdot\mathbf{V}' = 0, \tag{12}$$

where $P' = p' + (\mathbf{B}_0\mathbf{B}')/4\pi$, $\mathbf{V}_0$ is the unperturbed flow velocity, and $\mathbf{B}_0$ is the unperturbed magnetic field. The external force is not explicitly included in this system. It emerges in the expression of the unperturbed flow (6). Such behavior is typical for the parametric action.

The set of equations (9)–(12) has both spatial and temporal inhomogeneities due to Eq. (6). Therefore, the normal modal theory leads to the set of partial differential equations, which includes both spatial and temporal derivatives. Investigation of such a system is a complicated matter. However, Eqs. (9)–(12) retain only the temporal inhomogeneity in the comoving coordinates of the unperturbed flow,

$$x_1 = x + \frac{\alpha}{\rho\omega_0^2}\cos(\omega_0 t)y, \quad y_1 = y, \quad z_1 = z, \quad t_1 = t. \tag{13}$$

Then the Fourier expansion with respect to the spatial coordinates is possible,

$$\phi' = \int dk_{x_1}dk_{y_1}dk_{z_1}\phi(k_{x_1},k_{y_1},k_{z_1},t_1) \times \exp[i(k_{x_1}x_1 + k_{y_1}y_1 + k_{z_1}z_1)]. \tag{14}$$

Consequently, one can obtain the system with the time derivatives only,

$$\frac{\partial b_y}{\partial t_1} = iB_0 k_{x1} u_y, \tag{15}$$

$$\frac{\partial b_z}{\partial t_1} = iB_0 k_{x1} u_z, \tag{16}$$

$$k_{x1}b_x + \left[k_{y1} + \frac{\alpha}{\rho\omega_0^2}\cos(\omega_0 t_1)k_{x1}\right]b_y + k_{z1}b_z = 0, \tag{17}$$

$$\frac{\partial u_x}{\partial t_1} = -\frac{i}{\rho}k_{x1}P + \frac{iB_0}{4\pi\rho}k_{x1}b_x - V_{xy}u_y, \tag{18}$$

$$\frac{\partial u_y}{\partial t_1} = -\frac{i}{\rho}\left[k_{y1} + \frac{\alpha}{\rho\omega_0^2}\cos(\omega_0 t_1)k_{x1}\right]P + \frac{iB_0}{4\pi\rho}k_{x1}b_y, \tag{19}$$

$$\frac{\partial u_z}{\partial t_1} = -\frac{i}{\rho}k_{z1}P + \frac{iB_0}{4\pi\rho}k_{x1}b_z, \tag{20}$$

$$k_{x1}u_x + \left[k_{y1} + \frac{\alpha}{\rho\omega_0^2}\cos(\omega_0 t_1)k_{x1}\right]u_y + k_{z1}u_z = 0, \tag{21}$$

where $P = p + B_0 b_x/4\pi$. The system (15)–(21) describes the evolution of the amplitudes of Fourier harmonics introduced in Eq. (14). The wave numbers of the Fourier harmonics in the initial frame can be derived from Eqs. (13) and (14),

$$k_x = k_{x1}, \quad K_y = k_{y1} + \frac{\alpha}{\rho\omega_0^2}\cos(\omega_0 t_1)k_{x1}, \quad k_z = k_{z1}. \tag{22}$$

Note that the wave number along the shear axis (Y) is the periodic function of time.

In the absence of the mean (equilibrium) flow V_x and pressure perturbation p, Eqs. (15)–(21) describe the evolution of classical Alfven waves, which satisfy the conditions

$$(\mathbf{B}'\mathbf{B}_0) = 0, \quad (\mathbf{V}'\mathbf{B}_0) = 0. \tag{23}$$

However, the retention of pressure perturbations $\hat{p}$ in Eqs. (15)–(21) disrupts the conditions (23). In addition, the unperturbed flow (6) disrupts again the conditions (23) and leads to the emergence of another type of wave. Therefore, we argue that not only the classical Alfven waves, but also slow MHD waves are described by Eqs. (15)–(21). Eliminating P, b_x, v_x in Eqs. (15)–(21), we obtain the following system:

$$\frac{\partial b_y}{\partial t} = iB_0 k_x u_y, \tag{24}$$

$$\frac{\partial b_z}{\partial t} = iB_0 k_x u_z, \tag{25}$$

$$\frac{\partial u_y}{\partial t} = \frac{iB_0 k_x}{4\pi\rho} b_y + \frac{2V_{xy} k_x K_y}{k_x^2 + K_y^2 + k_z^2} u_y, \tag{26}$$

$$\frac{\partial u_z}{\partial t} = \frac{iB_0 k_x}{4\pi\rho} b_z + \frac{2V_{xy} k_x k_z}{k_x^2 + K_y^2 + k_z^2} u_y. \tag{27}$$

Alfven and slow MHD waves may be distinguished by their polarization. Alfven waves are polarized in the plane normal to the unperturbed magnetic field and wave vector, i.e., $(\mathbf{kB}') = 0$, $(\mathbf{B}'\mathbf{B}_0) = 0$. On the other hand, slow MHD waves are polarized in any plane satisfying the conditions: $(\mathbf{kB}') = 0$, $(\mathbf{kV}') = 0$. These different polarizations allow the separation of the solutions for Alfven and slow MHD waves. However, these waves acquire properties different from the classical ones in periodic shear flows.

The following two second-order differential equations can be obtained from the system (24)–(27):

$$\frac{\partial^2 b_y}{\partial t^2} - \frac{2V_{xy} k_x K_y}{k_x^2 + K_y^2 + k_z^2} \frac{\partial b_y}{\partial t} + \frac{k_x^2 B_0^2}{4\pi\rho} b_y = 0, \tag{28}$$

$$\frac{\partial^2 b_z}{\partial t^2} + \frac{k_x^2 B_0^2}{4\pi\rho} b_z = \frac{2V_{xy} k_x k_z}{k_x^2 + k_y^2 + k_z^2} \frac{\partial b_y}{\partial t}. \tag{29}$$

Using the substitution

$$b_y = \hat{b}_y(t) \exp\left(\int V_{xy} \frac{k_x K_y}{k_x^2 + K_y^2 + k_z^2} dt \right),$$

Eq. (28) leads to the Hill's equation:

$$\frac{\partial^2 \hat{b}_y}{\partial t^2} + \left[\frac{k_x^2 B_0^2}{4\pi\rho} + \frac{k_x K_y \dot{V}_{xy}}{k_x^2 + K_y^2 + k_z^2} + \frac{k_x V_{xy} \dot{K}_y}{k_x^2 + K_y^2 + k_z^2} + \frac{V_{xy}^2 k_x^2 K_y^2}{(k_x^2 + K_y^2 + k_z^2)^2} \right] \hat{b}_y = 0. \tag{30}$$

Analytical study of this equation can be simplified using the assumption of the weak inhomogeneity of the external force:

$$\alpha \ll \rho \omega_0^2. \tag{31}$$

Then the Hill type equation (30) turns into the equation of Mathieu,

$$\frac{\partial^2 \hat{b}_y}{\partial t^2} + \left[\frac{k_x^2 B_0^2}{4\pi\rho} + \gamma \cos(\omega_0 t) \right] \hat{b}_y = 0, \tag{32}$$

where

$$\gamma = \frac{\alpha}{\rho} \frac{k_x k_y}{k_x^2 + k_y^2 + k_z^2}.$$

This equation is well known in the theoretical mechanics and governs oscillations of a mathematical pendulum with a periodically varying length. When the frequency of the variation of the eigenfrequency is twice as large as the eigenfrequency of the pendulum itself, then the oscillations grow in amplitude exponentially. In the case of a magnetized medium, there are many wave modes with different frequencies and wave numbers satisfying the dispersion relation. The harmonics falling in resonance with the external force grow in amplitudes. Therefore the resonant conditions are imposed on the wave numbers. The main resonance in Eq. (32) occurs when

$$\frac{k_x B_0}{\sqrt{4\pi\rho}} = \frac{\omega_0}{2}, \tag{33}$$

i.e., the external energy transforms into the energy of the harmonics with k_x. The width of the resonant interval may be expressed as follows [29]:

$$\left| \frac{k_x B_0}{\sqrt{4\pi\rho}} - \frac{\omega_0}{2} \right| < \left| \frac{\gamma}{\omega_0} \right|. \tag{34}$$

The resonant solution of Eq. (32) may be obtained in the following analytical form [29]:

$$\hat{b}_y = b_0 \exp\left[\frac{\gamma}{2\omega_0} t \right] \left[\cos \frac{\omega_0}{2} t - \sin \frac{\omega_0}{2} t \right] = \sqrt{2} b_0 e^{(\gamma/2\omega_0 t)} \cos\left[\frac{\omega_0}{2} t + \frac{\pi}{4} \right], \tag{35}$$

where $b_0 = \hat{b}_y(0)$. This solution describes the evolution of the amplitudes of the Alfven waves polarized in the $Y0Z$ plane with the initial condition $\hat{b}_x = 0$, i.e., $K_y(0) b_y(0) + k_z b_z(0) = 0$, and slow MHD waves polarized in any plane satisfying the condition $k_x b_x(0) + K_y(0) b_y(0) + k_z b_z(0) = 0$. The waves with $K_y(0) \sim k_x > k_z$ have maximal growth rates. The waves change the polarization due to the temporal behavior of the wave vector (22). It generates the X component of the Alfven waves and breaks down the classical condition $(\mathbf{B}'\mathbf{B}_0) = 0$. Two main reasons may be responsible for this process: either Alfven waves lose their classical properties in the periodic shear flows or they are partially turned into the slow MHD waves.

Equations (34) and (35) show that the width of the resonant interval and the growth rate of the resonant waves depend on the shear rate α of the velocity flow generated by the external action. It has been known from the theory of mathematical pendulum that parametric resonance takes place near the frequencies $n\omega_0/2$. However, the growth rate is maximal at $n = 1$ and decreases quickly with the increase of n. Therefore, only the harmonics with frequency $\omega_0/2$ have significant growth rates.

Analytical solution (35) was found for the weakly inhomogeneous external force. However, it is necessary to provide the numerical simulation of the system (24)–(27) for arbitrary shear rates.

A. Numerical simulation

For the purpose of simplification of the numerical analysis, it is convenient to work with the dimensionless equations. Variables and functions have been made dimensionless as follows:

$$b_1 \equiv \frac{b_x}{B_0},\ \ b_2 \equiv \frac{b_y}{B_0},\ \ b_3 \equiv \frac{b_z}{B_0},\ \ v_1 \equiv \frac{iu_x t_0}{R_0},\ \ v_2 \equiv \frac{iu_y t_0}{R_0},$$

$$v_3 \equiv \frac{iu_z t_0}{R_0},\ \ V_{12} \equiv V_{xy} t_0,\ \ k_1 \equiv k_x R_0,\ \ k_2 \equiv k_y R_0,$$

$$k_3 \equiv k_z R_0,\ \ \tau \equiv t/t_0,\ \ \omega \equiv \omega_0 t_0,\ \ V_A \equiv \frac{B_0}{\sqrt{4\pi\rho}},$$

$$R_0 \equiv V_A t_0,\ \ a \equiv \frac{\alpha t_0^2}{\rho_0}, \tag{36}$$

where t_0 is the period of an external force.

By assigning the corresponding initial polarization one can look for the solutions of the Alfven and slow MHD waves separately. We suppose that initially Alfven waves are polarized in the $Y0Z$ plane and slow waves are polarized near the $X0Y$ plane.

Temporal evolution of components and total energy of resonant Alfven waves is presented in Fig. 1 for the small shear rate of the external action. The initial conditions are

$$b_1(0)=0,\ \ b_2(0)=0.1,\ \ b_3(0)=0.3,\ \ v_1(0)=0,$$

$$v_2(0)=-0.1,\ \ v_3(0)=-0.3.$$

Initially, Alfven waves are polarized in the $Y0Z$ plane. It is clearly seen that resonant harmonics have periods that are twice as large as the period of the external action. This figure illustrates that the Y component of the magnetic field is growing exponentially, whereas the Z component is not. Note that the longitudinal X component is generated after the initial phase. It is exponentially growing and breaks down the classical properties of the Alfven waves $(\boldsymbol{B}'\boldsymbol{B}_0)=0$. As a consequence, the polarization plane turns towards the direction of the $X0Y$ plane. As already noted, two possibilities may cause this process: either Alfven waves lose their classical properties or they partially transform into the slow MHD waves in the forced medium. Consideration of the compressibility will offer a clearer view on the understanding of this process. The normalized total spectral energy of the resonant harmonics in the **k** space is calculated using the following equation:

$$\frac{E}{E_0} = \frac{1}{2}(|v_1|^2+|v_2|^2+|v_3|^2)+(|b_1|^2+|b_2|^2+|b_3|^2), \tag{37}$$

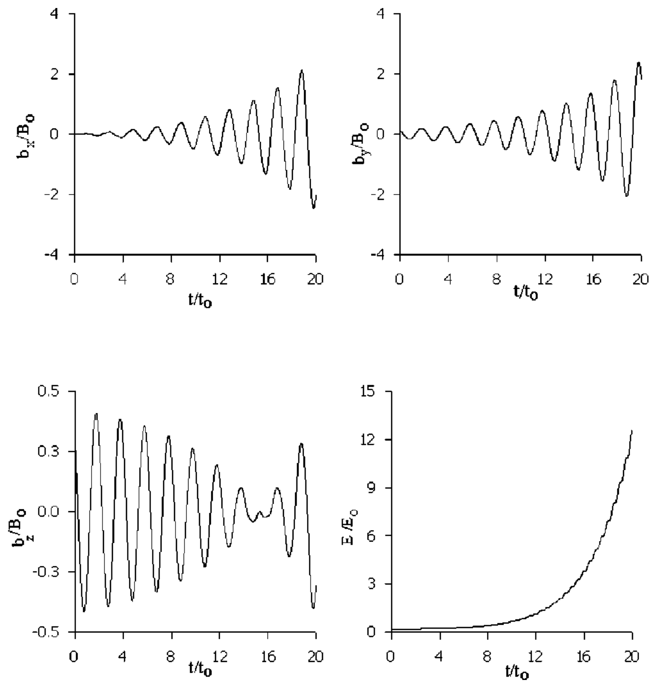

FIG. 1. Temporal evolution of the velocity and magnetic field components and the normalized total energy of resonant Alfven waves. Time is normalized by the period of the external action t_0. It is clearly seen that the period of waves is twice as large as t_0. Note the generation of the X component of the perturbations which was absent at the initial stage. Here, $k_1=3.14$, $K_2(0)=3.3$, $k_3=-1.1$, $a=0.6$, $E_0=B_0^2/4\pi$.

where $E_0=B_0^2/4\pi$, the first is the kinetic term and the second is the magnetic one. As is seen on the plot, the energy of the perturbations is exponentially growing. Consequently, it is safe to say that perturbations take the energy from the background flow or, more properly, energy of the external force transforms into the magnetic energy of the system.

Figure 2 illustrates the resonant solutions of the slow MHD waves for the same shear rate as in Fig. 1. The initial conditions are

$$b_1(0)=-0.063,\ \ b_2(0)=0.1,\ \ b_3(0)=3,$$

$$v_1(0)=0.06,\ \ v_2(0)=-0.1,\ \ v_3(0)=-3.$$

The properties of slow waves are the same as the Alfven ones despite the fact that the polarization does not change the plane during the wave evolution.

To gain a complete understanding of the external action it is useful to conduct the numerical simulations for the larger shear rate α. Figures 3 and 4 show the solutions for the Alfven and slow waves in the case when α is three times larger compared to Figs. 1 and 2. The initial conditions are

$$b_1(0)=0,\ \ b_2(0)=0.1,\ \ b_3(0)=0.3,\ \ v_1(0)=0,$$

$$v_2(0)=-0.1,\ \ v_3(0)=-0.3$$

for Fig. 3 and

$$b_1(0)=0.92,\ \ b_2(0)=0.1,\ \ b_3(0)=3,$$

$$v_1(0)=-0.92,\ \ v_2(0)=-0.1,\ \ v_3(0)=-3$$

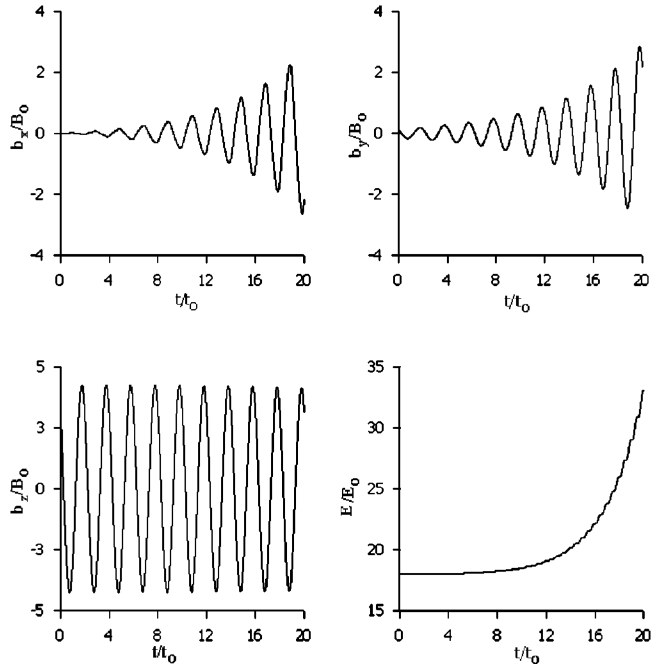

FIG. 2. Temporal evolution of the velocity and magnetic field components and the normalized total energy of resonant slow MHD waves. They behave similarly to the Alfven waves. Here, $k_1 = 3.14$, $K_2(0) = 5.3$, $k_3 = -0.11$, $a = 0.6$, $E_0 = B_0^2/4\pi$.

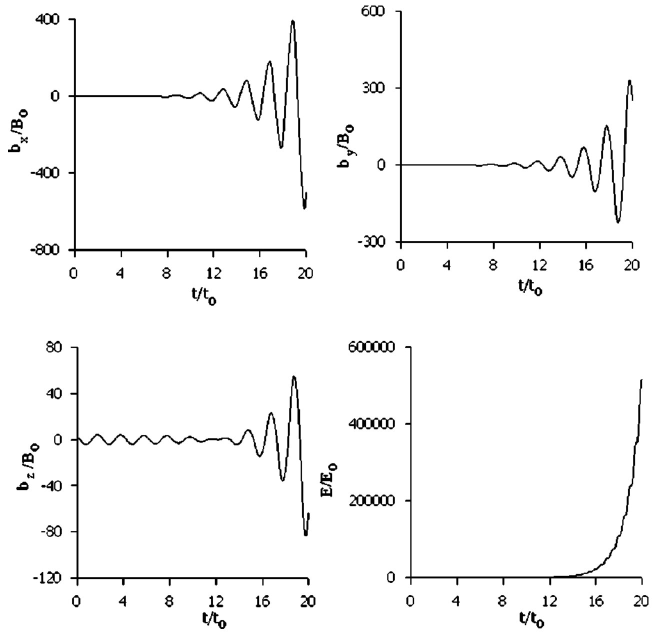

FIG. 4. Temporal evolution of the velocity and magnetic field components and the normalized total energy of resonant slow MHD waves for the larger shear rate. Here, $k_1 = 3.14$, $K_2(0) = 5.9$, $k_3 = -1.3$, $a = 1.8$, $E_0 = B_0^2/4\pi$.

for Fig. 4. Referring to these figures, the shear rate is a very sensitive parameter for the resonance: the waves undergo the strongest amplification in comparison to the above cases.

Analytical and numerical study in the linear approximation gives indications of the parametric resonance for the Alfven and slow MHD waves due to the external action. Resonant harmonics grow exponentially in time, and consequently nonlinear interaction cannot be neglected after a lapse of time. The study of the complete nonlinear dynamics of the parametric resonance is an intricate problem. Hence, we provide only a formal nonlinear analysis and anticipate further development in the future.

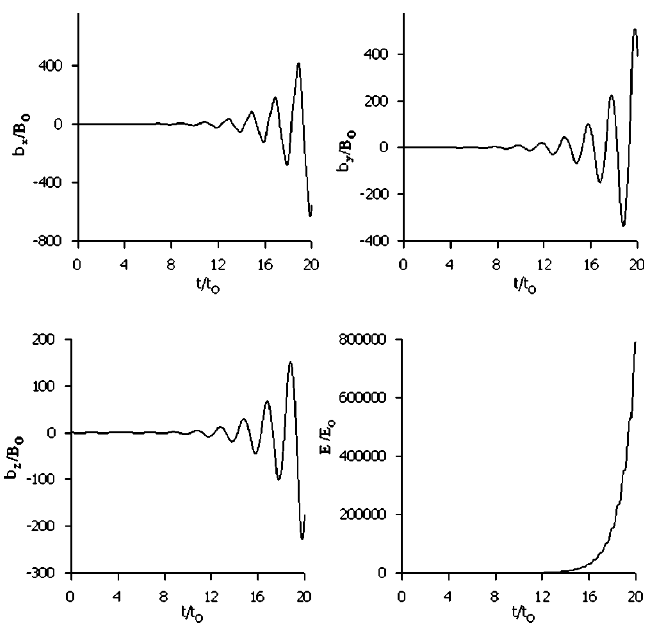

FIG. 3. Temporal evolution of the velocity and magnetic field components and the normalized total energy of resonant Alfven waves for the larger shear rate. Here, $k_1 = 3.14$, $K_2(0) = 3.9$, $k_3 = -1.3$, $a = 1.8$, $E_0 = B_0^2/4\pi$. The strongest growth of the amplitudes is clearly seen.

B. Formal nonlinear analysis

The properties of nonlinear MHD waves have been studied intensively [30–33]. Generally, nonlinear interaction generates the second and higher harmonics and leads to the frequency change of the initial resonant waves. Consequently, the harmonics are forced out from resonance. Therefore, the amplitudes of the resonant waves decrease, which means that there is a redistribution of the energy of the resonant harmonics to the higher ones. As a result the frequency of resonant waves in Eq. (32) becomes amplitude dependent,

$$V_A^2 k_x^2 \rightarrow V_A^2 k_x^2 (1 + \beta b_y^2), \tag{38}$$

where β is the coefficient of nonlinearity. This additional term introduces a nonlinear term of the Schrödinger equation type in the Eq. (32). Using this formal substitution Mathieu's equation (32), which governs the linear parametric resonance, leads to the equation governing the nonlinear dynamics of the system,

$$\frac{\partial^2 b_y}{\partial t^2} + [V_A^2 k_x^2 (1 + \beta b_y^2) + \gamma \cos(\omega_0 t)] b_y = 0. \tag{39}$$

Introducing the new function $V \equiv \partial b_y / \partial t$, Eq. (39) may be expressed in the form of the following two differential equations:

$$\frac{\partial b_y}{\partial t} = V, \tag{40}$$

$$\frac{\partial V}{\partial t} = -\frac{\omega_0^2}{4}\left(1+4\frac{\gamma}{\omega_0^2}\cos(\omega_0 t)\right)b_y - \beta\frac{\omega_0^2}{4}b_y^3. \quad (41)$$

We look for the solution of the system (40) and (41) in the following form:

$$b_y = 2A(t)\cos\left[\frac{\omega_0}{2}t + \varphi(t)\right]. \quad (42)$$

Substitution of Eq. (42) into Eq. (40) leads to the following equation:

$$2\dot{A}\cos\left[\frac{\omega_0}{2}t+\varphi(t)\right] - 2A\left(\frac{\omega_0}{2}+\dot{\varphi}\right)\sin\left[\frac{\omega_0}{2}t+\varphi(t)\right] = V. \quad (43)$$

Equation (42) substitutes the two variables A and φ. Thus one can choose an arbitrary relation between these new variables. For simplicity we consider the following relation:

$$\dot{A}\cos\left[\frac{\omega_0}{2}t+\varphi(t)\right] - A\dot{\varphi}\sin\left[\frac{\omega_0}{2}t+\varphi(t)\right] = 0. \quad (44)$$

Then the expression for V takes the form

$$V = -A\omega_0\sin\left[\frac{\omega_0}{2}t+\varphi(t)\right]. \quad (45)$$

Substitution of Eq. (45) into Eq. (41) and consequent averaging over the period leads to the following equations for the amplitude and phase:

$$\dot{A} = \frac{\gamma}{2\omega_0}A\sin 2\varphi, \quad (46)$$

$$\dot{\varphi} = \frac{3\beta\omega_0}{4}A^2 + \frac{\gamma}{2\omega_0}\sin 2\varphi. \quad (47)$$

Hence, integration of Eq. (46) allows us to get the following equation for the amplitude:

$$A = C\exp\left(\frac{\gamma}{2\omega_0}\int \sin 2\varphi dt\right), \quad (48)$$

where C is a certain constant of integration. This equation indicates that the amplitude of a resonant harmonic undergoes periodical growth and decay due to the nonlinear interaction, which is expressed by the first term on the right-hand side of Eq. (47). The rate and the period of the amplitude variation depend on γ, the coefficient of nonlinearity β, and the initial amplitude of the magnetic field.

The numerical solution of the differential equations (40) and (41) is presented in Fig. 5 with the following conditions:

$$\beta B_0^2 = 0.005, \quad \frac{\gamma}{\omega_0^2} = 0.05, \quad V(0) = 0, \quad \frac{b_y(0)}{B_0} = 0.05.$$

Time is normalized by the period of the external force t_0. Further inspection of Fig. 5 shows that the period of the amplitude variation is much larger than the period of the resonant waves.

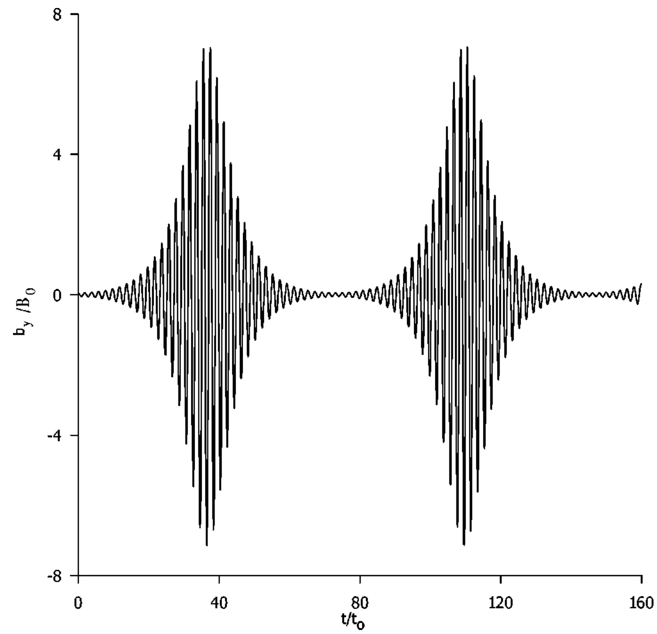

FIG. 5. Nonlinear modulation of the resonant Alfven waves amplitudes. The period of the modulation is much larger than the period of the resonant waves. Here, $\beta B_0^2 = 0.005$, $\gamma/\omega_0^2 = 0.05$, $V(0) = 0$, $b_y(0)/B_0 = 0.05$.

Hence we can imagine the process of the external action: the force strengthens certain harmonics with appropriate wavelengths at the initial linear stage. When amplitudes of the harmonics reach the high values, the nonlinear interaction turns on and leads to the redistribution of the energy from the resonant harmonics to the higher frequency ones. Consequently, amplitudes of the resonant harmonics are decreased, creating again the conditions for the linear parametric action. As a result, the repetitive amplification of the perturbations leads to the long-term modulation of their amplitudes. An explicit example of this process is the nonlinear oscillation of the mathematical pendulum with periodically varying lengths in theoretical mechanics. This mechanism may be responsible for the explanation of the Maunder minimum in the solar activity.

III. DISCUSSION

Recently observed uncertainties of the magnetic field behavior in the Sun, binary stars, and galaxies set a stage for modifying the classical dynamo theory or developing new theories. The bisymmetric structure of galactic magnetic fields that closely follow spiral arms cannot be explained by classical turbulent dynamo theory. Therefore, an explanation of observations necessitates the consideration of density waves [5,6]. These waves may be coupled with dynamo ones if their frequencies satisfy the relation $2\div 1$. However, this condition will only be satisfied fortuitously [34]. Furthermore, the turbulent dynamo theory encounters problems trying to explain the solar magnetic cycles [28]. The different character of the magnetic fields in binary stars also demands further investigation [27]. As a consequence, an explanation of the magnetic field behavior in many astrophysical objects requires an overview of the amplification process.

The energy supply for the magnetic field in the classical dynamo theory is the energy of the turbulent motion in differentially rotating objects. However, the role of external sources in the magnetic field amplification process is not completely understood.

An influence of an external electromagnetic force on plasma has been studied extensively [1]. A question arises: what happens if a nonelectromagnetic external action takes place? It is obvious that a nonelectromagnetic force cannot directly affect magnetic waves. However, indirect influence may be assumed in the form of a parametric action.

The nonelectromagnetic inhomogeneous periodic force directed along the magnetic field produces periodical shear motions of plasma leaving field lines unaltered. The stability of such plasma is studied in the present paper. MHD equations, written in the comoving coordinates of the unperturbed flow, lead to the Hill's equation for an incompressible medium. In case of the weakly inhomogeneous external action the Hill's equation turns into the Mathieu's equation (32), which describes the parametric resonance in the theoretical physics. As expected intuitively, periodical variation of the velocity shear generates the Alfven waves with exponentially growing amplitudes. Contrary to a single mathematical pendulum, there are many wave modes in the plasma with different frequencies and wavelengths satisfying the dispersion relation of Alfven waves. As derived in Eq. (32), parametric resonance requires the condition

$$\frac{k_x B_0}{\sqrt{4\pi\rho}} = \frac{\omega_0}{2}.$$

This equation expresses the fact that the period of the resonant harmonics is twice as large as the period of the external action. As a result, only harmonics with certain wavelengths satisfy the resonant conditions. To put it differently, the external force transmits the energy to the selected harmonics. Amplitudes of these harmonics grow exponentially in time (see Figs. 1–4). The growth rate and the resonant interval depend on the velocity shear rate generated by the external action.

As already noted, the wave vector of Alfven waves undergoes a periodic drift in **k** space. As a result, the plane of the polarization undergoes a periodical variation which generates the longitudinal component perturbing the classical condition $(\mathbf{B}'\mathbf{B}_0) = 0$. Two main mechanisms can be responsible for this phenomenon: either Alfven waves are modified in the forced medium or they partially transform into slow MHD waves. We cannot prefer one of these two possibilities in an incompressible limit. It requires a study of a compressible medium.

The exponential growth of wave amplitudes indicates that the linear approximation loses its validity soon. Then a formal non-linear analysis shows periodical growth and decay of the resonant harmonics. This amplitude modulation is a result of the nonlinear interaction between resonant harmonics that produces overtones of a second and higher order: $2\omega, 3\omega, \ldots$. As a result, the energy of the resonant harmonics is redistributed between the energies of higher-order harmonics. Consequently, amplitudes of the resonant harmonics are reduced. They became smaller and the linear mechanism of parametric amplification turns on again and leads to the perturbation growth. The period of the modulation is larger than the period of the resonant harmonics and depends on the velocity shear rate generated by the external force and the coefficient of the nonlinearity (see Fig. 5).

The mechanism of magnetic field amplification presented here may be applied to many astrophysical objects. We present several examples of external action, which can generate parametric resonance.

A. The Sun

It has been shown that gravitational action of the planets correlates with the solar activity cycles [35]. However, there was no physical mechanism explaining such correlation. Recently, Zaqarashvili [3] has shown that in the case of elliptical motion of the Sun about the mass center of the solar system (which is the result of the planetary gravity), noninertial force $\rho\mathbf{r}\times\dot{\mathbf{\Omega}}$ influences the solar plasma. Angular velocity Ω is the periodic function of time for eccentric orbits. Therefore, this force is periodic and inhomogeneous. Hence it generates periodic shear of the solar internal rotation with the period of $\sim$11 years (it corresponds to the period of Jupiter, which is the biggest planet in the solar system). Then, the parametric amplification of Alfven waves with the period of 22 years occurs in the solar interior. This result is very intriguing because the solar magnetic field actually has the same period. Nonlinear long-term modulation of amplitudes found in the present paper may be responsible for the generation of the Maunder minimum in the solar magnetic activity. This theory may be used in searching for extrasolar planets as well: if a star exhibits a periodical variation in its chromospheric activity then one can suggest the existence of the planet with the same orbital period.

B. Binary stars

Several years ago, Schrijver and Zwaan [27] have shown based on the observational data that the chromospheric activity in binary systems is substantially higher in comparison to the single stars of the same spectral classes and rotation periods. They have concluded that gravity of a companion affects the differential rotation of a primary star and may be responsible for a so-called overactivity. However, they have not proposed any physical mechanism explaining the strengthening of the magnetic field. In this regard the parametric resonance presented in this paper offers the greatest promise. Any little asymmetry in a binary system (eccentric orbit or inclination of the rotation axis) causes a periodic shearing of the internal rotation and thus amplification of the magnetic field.

C. Galaxies

It seems reasonable to expect that the parametric amplification of the magnetic field can occur in interacting galaxies. This suggestion is supported by the observed fact that interacting galaxies exhibit rather strong magnetic fields [36–38]. However, uncertainties in the measurements of the orbital parameters complicate the evaluation of the growth rate of magnetic fields. On the other hand, magnetic field amplification may occur at the expense of density waves in spiral galaxies. Density waves that propagate in spiral galaxies produce a periodic shear in the galactic rotation because of the angular momentum transfer. Considering the spiral density waves as an external force which produces periodic shear motions, one can find the parametric amplification of Alfven waves. In other words, we can consider the density wave, which is a consequence of gravitational and rotational actions, in a parametric form in ideal MHD equations. As a result, Alfven waves grow exponentially. It can be said with confidence that the energy of density waves is transformed into the energy of Alfven waves. In other words, waves of one type change the properties of a medium and offer a channel for the transmission of energy to the waves of another type. Indeed, this phenomenon may be described in the framework of the linear theory, despite its intrinsic nonlinear character. We believe in the development of this method in the future. Let us evaluate the growth rate of perturbations in this case. Density waves cause the following shear in the rotation of the galaxies:

$$V_x = \frac{\alpha}{\rho\omega_d}\sin(\omega_d t)y,$$

where ω_d and α are the frequency and the strength of the density waves. The X axis is directed along the rotation of the galaxy and the Y axis has radial direction. The growth rate depends on the amplitude of density waves. The period of density waves is considered to be equal to the period of the galactic rotation [39]. Taking $\alpha/\rho \sim \omega_d^2$ during the lifetime of the Milky Way, which corresponds to ~ 50 rotations, amplitudes of the Alfven waves are amplified by the following factor: $\approx 10^{21}$. Despite a very rough estimation, without taking into account nonlinear interactions, ambipoliar diffusion, and other damping processes, an extreme growth suggests the necessity of a further detailed study.

IV. CONCLUSION

A mechanism of a magnetic field amplification is outlined and discussed within the ideal MHD theory. It is shown that an external nonelectromagnetic inhomogeneous periodic force is able to generate resonant Alfven waves in a magnetized medium. During the process of the parametric resonance energy of the external nonelectromagnetic driving force transforms to the energy of the generated Alfven wave harmonics with definite wavelengths. The period of the resonant harmonic is twice as large as the period of the external force. The growth rate of the Alfven waves depends on the shear rate of the flow velocity, which originates from the external action. Furthermore, the nonlinear interaction between the resonant harmonics produces long-term modulation of their amplitudes.

The presented mechanism of energy transformation from an external nonelectromagnetic force into magnetic oscillations of the system can be used in the understanding of the physical basis of gravitational action on a magnetized medium. The possible application of this mechanism in the explanation of the magnetic field behavior in the Sun, binary stars, and spiral galaxies is briefly discussed.

Future consideration of the compressibility will provide a more detailed insight into the nature of this phenomenon.

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